Lecture 5: Inflationary Perturbations and the CMB

Graduate Course in Astroparticles and Cosmology

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Plan of the lecture

1. Inflation as initial-condition physics
2. Gauge-invariant scalar perturbations
3. The Sasaki–Mukhanov formalism
4. Tensor perturbations
5. Models and observational interpretation

Suggested references and resources

Textbooks and lecture notes

- V. Mukhanov – *Physical Foundations of Cosmology*
- S. Weinberg – *Cosmology*
- D. Baumann – *TASI Lectures on Inflation*
- Dodelson and Schmidt – *Modern Cosmology*

Original and classic papers

- Mukhanov and Chibisov – quantum origin of density perturbations
- Bardeen, Steinhardt and Turner – gauge-invariant perturbations from inflation
- Sasaki and Mukhanov – canonical variable and quantization of scalar perturbations

From the background Universe to perturbations

The previous lectures established the homogeneous FLRW background, the thermal history of the Universe, and the photon–baryon plasma before recombination.

The purpose of this lecture is to derive the initial conditions for that plasma from an inflationary phase. The central chain of reasoning is

$$\text{quantum fluctuations during inflation} \to \mathcal{R}(\mathbf{k}) \to \Theta(\hat{\mathbf{n}}) \to C_\ell\,. \tag{1}$$

Physical message

Inflation provides the primordial random field. Linear cosmological perturbation theory propagates this field to recombination. The CMB angular power spectrum measures its statistical properties.

Horizon and flatness as dynamical problems

In a decelerating Universe, the comoving Hubble radius grows,

$$(aH)^{-1} \quad \text{increases for} \quad \ddot{a} < 0. \tag{2}$$

Causally disconnected regions of the last-scattering surface appear to have had no time to thermalize.

Inflation reverses this behavior:

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0 \qquad \Longleftrightarrow \qquad \ddot{a} > 0. \tag{3}$$

A sufficiently long period of accelerated expansion allows the observable Universe to originate from one small causal domain.

Key consequence

Modes observed in the CMB were initially deep inside the Hubble radius, where the notion of a quantum vacuum is well defined.

Horizon problem

At recombination (t_*), the physical particle horizon is

$$d_{\rm hor}^{\rm phys}(t_*) = a(t_*) \int_0^{t_*} \frac{{\rm d}t}{a(t)} . \tag{4}$$

This corresponds to an angular scale on the sky of order $\theta_{\rm hor} \sim 1^\circ$.

Observation

The CMB temperature is uniform across the entire sky,

$$\Theta \equiv \frac{\Delta T}{T} \sim 10^{-5} , \tag{5}$$

even between regions separated by $\gg 1^\circ$.

Puzzle

In a decelerating Universe, regions separated by more than $\theta_{\rm hor}$ were never in causal contact before recombination, so no mechanism could have equilibrated their temperatures.

Flatness problem

The Friedmann equation can be written as

$$|\Omega - 1| = \frac{|K|}{(aH)^2}, \tag{6}$$

where K denotes the spatial-curvature parameter, not the Fourier wavenumber k. In a decelerating Universe ($\ddot{a} < 0$),

$$(aH)^{-1} \text{ grows with time} \quad \Rightarrow \quad |\Omega - 1| \text{ increases.} \tag{7}$$

Observation

Today,

$$\Omega \simeq 1 \quad \text{(spatially nearly flat Universe).} \tag{8}$$

At early times,

$$|\Omega - 1|_{\rm early} \ll 10^{-60}, \tag{9}$$

i.e. the initial curvature must have been tuned extraordinarily close to zero.

Monopole problem

Grand Unified Theories predict heavy topological relics (magnetic monopoles) produced during symmetry-breaking phase transitions.

A typical estimate gives a relic abundance

$$n_M \sim n_\gamma \quad \text{at production.} \tag{10}$$

After standard expansion, $\Omega_M \gg 1$, so monopoles would dominate the energy density of the Universe.

Observation

Relic number density scales as $n_M \propto a^{-3}$, so monopoles cannot be efficiently diluted in standard expansion. Why are these relics absent or extremely diluted if high-energy phase transitions occurred in the early Universe?

Comoving scales during inflation

For a Fourier mode with comoving wavenumber k, the comparison scale is the comoving Hubble radius. A mode is

$$\text{sub-horizon if } k \gg aH, \qquad \text{super-horizon if } k \ll aH. \tag{11}$$

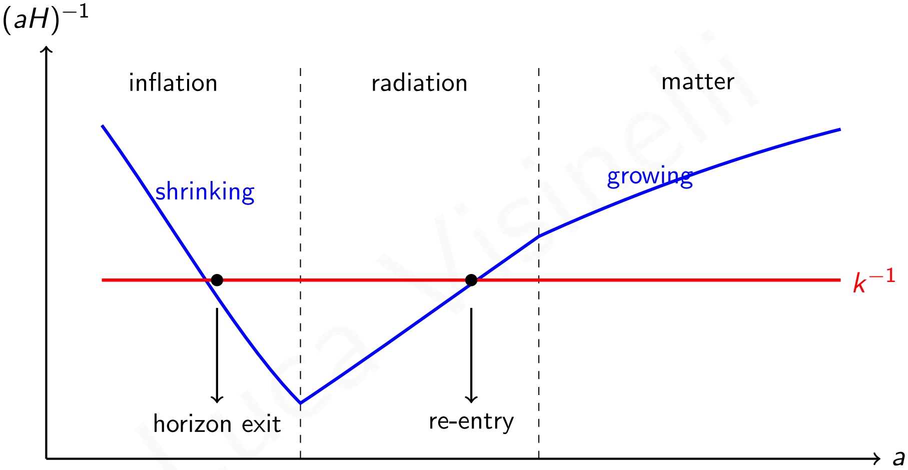
Horizon exit occurs when

$$k = aH. \tag{12}$$

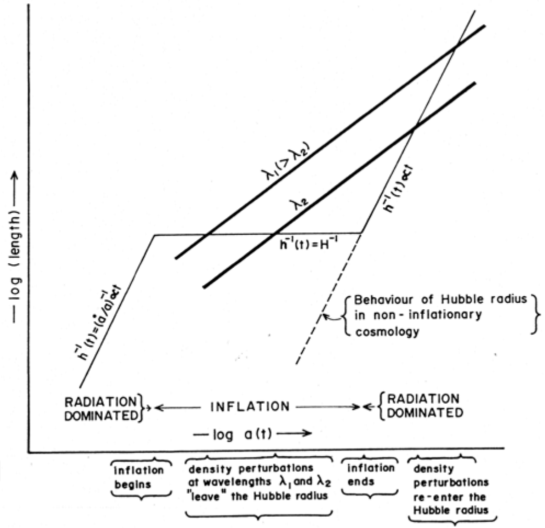
Key idea

Causal physics operates only on sub-horizon scales ($k \gg aH$).

Evolution of the comoving Hubble radius



During inflation, $(aH)^{-1}$ decreases, so modes exit the Hubble radius. After inflation, $(aH)^{-1}$ grows again, so the same modes re-enter during radiation or matter domination.



https://ned.ipac.caltech.edu/

Single-field inflationary background

Consider a canonical scalar field minimally coupled to gravity,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{13}$$

where $M_{\rm Pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass. The homogeneous energy density and pressure are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{14}$$

The background equations are

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right), \tag{15}$$

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{\rm Pl}^2}, \tag{16}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi}. \tag{17}$$

Slow-roll parameters

Inflation requires the kinetic energy of the inflaton to remain subdominant. It is useful to define the Hubble slow-roll parameter

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{\rm Pl}^2 H^2}. \tag{18}$$

Accelerated expansion is equivalent to

$$\ddot{a} > 0 \qquad \Longleftrightarrow \qquad \epsilon_H < 1. \tag{19}$$

In the potential slow-roll approximation,

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \qquad \eta_V \equiv M_{\rm Pl}^2 \frac{V_{,\phi\phi}}{V}, \tag{20}$$

and

$$3H\dot{\phi} \simeq -V_{,\phi}, \qquad H^2 \simeq \frac{V}{3M_{\rm Pl}^2}. \tag{21}$$

Number of e-folds

The total amount of accelerated expansion is measured by

$$N \equiv \int_{t_i}^{t_f} H \, \mathrm{d}t = \ln \frac{a_f}{a_i}. \tag{22}$$

In slow roll,

$$N(\phi) \simeq \frac{1}{M_{\rm Pl}^2} \int_{\phi_f}^{\phi} \frac{V}{V_{,\phi}} \, \mathrm{d}\phi. \tag{23}$$

The observed CMB scales typically exited the Hubble radius roughly

$$N_* \simeq 50 - 60 \tag{24}$$

e-folds before the end of inflation, with the precise value depending on reheating and the post-inflationary expansion history.

Why $N_* \sim 50$–60?

The relevant quantity is the number of e-folds between horizon exit of a CMB mode and the end of inflation:

$$k = a_* H_* \,. \tag{25}$$

Relating this scale to today gives schematically

$$\frac{k}{a_0 H_0} = \frac{a_*}{a_{\rm end}} \cdot \frac{a_{\rm end}}{a_{\rm reh}} \cdot \frac{a_{\rm reh}}{a_0} \cdot \frac{H_*}{H_0}. \tag{26}$$

Taking logarithms,

$$N_* \equiv \ln \frac{a_{\rm end}}{a_*} \simeq 50\text{–}60 + (\text{corrections from reheating and energy scale}). \tag{27}$$

N_* counts how much the Universe expanded after a mode left the horizon. It depends on the inflationary energy scale, the reheating history, the subsequent radiation and matter evolution.

Scalar perturbations of the metric

At linear order, scalar perturbations of a spatially flat FLRW metric can be written in conformal time as

$$\mathrm{d}s^2 = a^2(\eta)\Big\{ -(1+2A)\mathrm{d}\eta^2 + 2\partial_i B\,\mathrm{d}x^i\mathrm{d}\eta + \left[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E\right]\mathrm{d}x^i\mathrm{d}x^j \Big\}. \tag{28}$$

The inflaton is similarly decomposed as

$$\phi(\eta,\mathbf{x}) = \bar{\phi}(\eta) + \delta\phi(\eta,\mathbf{x}). \tag{29}$$

Not all of these variables are physical: scalar perturbations transform under infinitesimal coordinate transformations.

Gauge-invariant curvature perturbation

The comoving curvature perturbation is the curvature perturbation on hypersurfaces orthogonal to the total momentum flow. With the scalar-metric convention used here,

$$\mathcal{R} = -\psi - \frac{\mathcal{H}}{\bar{\phi}'}\delta\phi. \tag{30}$$

Here a prime denotes $\mathrm{d}/\mathrm{d}\eta$ and $\mathcal{H} = a'/a$.

In comoving gauge, $\delta\phi = 0$, so

$$\mathcal{R} = -\psi. \tag{31}$$

In spatially flat gauge, $\psi = 0$, so

$$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi_{\mathrm{flat}}. \tag{32}$$

Why $\mathcal{R}$ is central

For adiabatic perturbations, $\mathcal{R}$ is conserved on super-horizon scales, connecting horizon exit during inflation to the CMB at recombination.

Conservation of $\mathcal{R}$ on large scales

For scalar perturbations, the evolution of $\mathcal{R}$ can be written schematically as

$$\mathcal{R}' = -\frac{\mathcal{H}}{\rho + p}\,\delta p_{\mathrm{n\text{-}ad}} + \mathcal{O}\left(\frac{k^2}{a^2H^2}\right), \tag{33}$$

where $\delta p_{\mathrm{n\text{-}ad}}$ is the non-adiabatic pressure perturbation.

For single-field slow-roll inflation,

$$\delta p_{\mathrm{n\text{-}ad}} \simeq 0, \qquad k \ll aH, \tag{34}$$

$$\mathcal{R}_k \simeq \text{constant} \qquad \text{on super-horizon scales.} \tag{35}$$

Consequence

The amplitude of each scalar mode can be computed at horizon exit and then propagated through cosmic history as an initial condition.

The canonical scalar degree of freedom

The scalar perturbations of the inflaton and metric are constrained variables. After solving the lapse and shift constraints, the physical scalar degree of freedom is described by the Mukhanov–Sasaki variable

$$v \equiv z\mathcal{R}, \qquad z \equiv \frac{a\bar{\phi}'}{\mathcal{H}} = a\frac{\dot{\bar{\phi}}}{H} = aM_{\rm Pl}\sqrt{2\epsilon_H}. \tag{36}$$

The quadratic action becomes

$$S^{(2)} = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3x \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]. \tag{37}$$

Interpretation

Scalar cosmological perturbations reduce to a canonically normalized field with a time-dependent effective mass in an expanding background.

Sasaki–Mukhanov equation

Expanding in Fourier modes,

$$v(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} v_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{38}$$

the equation of motion is

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0. \tag{39}$$

Fundamental equation for scalar perturbations in single-field inflation.

The two limiting regimes are:

$$k^2 \gg z''/z : \quad v_k'' + k^2 v_k \simeq 0, \quad \text{oscillatory quantum vacuum,} \tag{40}$$

$$k^2 \ll z''/z : \quad \mathcal{R}_k = v_k/z \simeq \text{constant}, \quad \text{classical curvature perturbation.} \tag{41}$$

Slow-roll form of the effective mass

In quasi-de Sitter expansion,

$$a(\eta) \simeq -\frac{1}{H\eta}, \qquad \eta < 0, \tag{42}$$

and the effective mass is approximately

$$\frac{z''}{z} \simeq \frac{\nu_s^2 - 1/4}{\eta^2}. \tag{43}$$

To leading order in slow roll,

$$\nu_s \simeq \frac{3}{2} + 3\epsilon_H - \eta_H, \tag{44}$$

where $\eta_H \equiv -\ddot{\phi}/(H\dot{\phi})$.

Quantization and the Bunch–Davies vacuum

Promote the canonical variable to an operator,

$$\hat{v}(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[v_k(\eta) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\eta) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{45}$$

with

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{q}). \tag{46}$$

Deep inside the horizon, spacetime curvature is negligible and the positive-frequency solution is selected:

$$v_k(\eta) \longrightarrow \frac{e^{-ik\eta}}{\sqrt{2k}}, \qquad k \gg aH. \tag{47}$$

This is the Bunch–Davies initial condition.

Physical meaning of the Bunch–Davies vacuum

The Bunch–Davies choice corresponds to selecting the vacuum state such that, at very early times,

$$v_k(\eta) \longrightarrow \frac{e^{-ik\eta}}{\sqrt{2k}}, \qquad k \gg aH. \tag{48}$$

Key idea

At short wavelengths ($k \gg aH$), spacetime curvature is negligible and each mode behaves like a harmonic oscillator in flat spacetime.

- Each Fourier mode is placed in its **minimum-energy (vacuum) state**.
- This is the natural generalization of the **Minkowski vacuum** to an expanding Universe.
- The vacuum is defined at early times, when all observable modes are deep inside the horizon.

Mode functions and freeze-out

For slowly varying ν_s, the solution is written in terms of Hankel functions,

$$v_k(\eta) = \frac{\sqrt{-\pi\eta}}{2} e^{i\pi(\nu_s+1/2)/2} H^{(1)}_{\nu_s}(-k\eta). \tag{49}$$

For exact de Sitter, $\nu_s = 3/2$, and

$$v_k(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \tag{50}$$

At late times, $|k\eta| \ll 1$, the curvature perturbation

$$\mathcal{R}_k = \frac{v_k}{z} \tag{51}$$

approaches a constant growing-mode amplitude.

Scalar power spectrum

The dimensionless power spectrum is defined by

$$\langle \mathcal{R}(\mathbf{k})\mathcal{R}^*(\mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')P_{\mathcal{R}}(k), \qquad \mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}P_{\mathcal{R}}(k). \quad (52)$$

Using $z = aM_{\rm Pl}\sqrt{2\epsilon_H}$, the slow-roll result is

$$\mathcal{P}_{\mathcal{R}}(k) \simeq \frac{1}{8\pi^2 M_{\rm Pl}^2}\frac{H^2}{\epsilon_H}\bigg|_{k=aH} \simeq \frac{1}{24\pi^2 M_{\rm Pl}^4}\frac{V}{\epsilon_V}\bigg|_{k=aH}. \quad (53)$$

Observations give approximately

$$\mathcal{P}_{\mathcal{R}}(k_*) \simeq 2.1\times 10^{-9}, \qquad k_* = 0.05\,{\rm Mpc}^{-1}. \quad (54)$$

Spectral tilt

The scalar spectral index is defined as

$$n_s - 1 \equiv \frac{\mathrm{d}\ln \mathcal{P}_{\mathcal{R}}}{\mathrm{d}\ln k}. \tag{55}$$

Since modes are evaluated near $k = aH$, derivatives with respect to $\ln k$ can be related to derivatives with respect to e-fold time. At leading order,

$$n_s - 1 \simeq -6\epsilon_V + 2\eta_V. \tag{56}$$

Exact scale invariance corresponds to $n_s = 1$. Observations find

$$n_s \simeq 0.965, \tag{57}$$

so the scalar spectrum is slightly red tilted.

Physical origin of the tilt

The Hubble parameter and slow-roll parameters evolve slowly during inflation, so different modes freeze out at slightly different amplitudes.

Gaussianity and the role of interactions

The quadratic action determines a Gaussian random field. In this approximation, all statistical information is contained in the two-point function.

Higher-order interactions generate connected higher-point functions, for example

$$\langle \mathcal{R}(\mathbf{k}_1)\mathcal{R}(\mathbf{k}_2)\mathcal{R}(\mathbf{k}_3)\rangle = (2\pi)^3 \delta^{(3)}\left(\sum_i \mathbf{k}_i\right) B_{\mathcal{R}}(k_1, k_2, k_3). \quad (58)$$

Single-field slow-roll inflation predicts small non-Gaussianity, parametrically of order slow-roll parameters.

Observational meaning

The near-Gaussianity of the CMB is not accidental: it reflects the weakness of interactions in the inflationary perturbation theory.

Primordial tensor modes

Tensor perturbations are transverse and traceless metric fluctuations,

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right], \tag{59}$$

with

$$\partial_i h_{ij} = 0, \qquad h^i{}_i = 0. \tag{60}$$

Expanding into two polarization states,

$$h_{ij}(\eta, \mathbf{x}) = \sum_{\lambda=+,\times} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} h_\lambda(\eta, \mathbf{k}) e^{\lambda}_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{61}$$

Each polarization behaves like a massless field in quasi-de Sitter spacetime.

Tensor spectrum and tensor-to-scalar ratio

The dimensionless tensor power spectrum is

$$\mathcal{P}_T(k) \simeq \frac{2}{\pi^2} \frac{H^2}{M_{\rm Pl}^2}\bigg|_{k=aH}. \tag{62}$$

The tensor-to-scalar ratio is

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\mathcal{R}} \simeq 16\epsilon_H. \tag{63}$$

The tensor tilt is

$$n_T \equiv \frac{\mathrm{d}\ln \mathcal{P}_T}{\mathrm{d}\ln k} \simeq -2\epsilon_H, \tag{64}$$

leading to the single-field consistency relation

$$r \simeq -8n_T\,. \tag{65}$$

Observational target

A detection of primordial tensor modes would directly measure the inflationary Hubble scale and the energy scale of inflation.

Energy scale of inflation

Combining the scalar amplitude with $r = 16\epsilon_H$ gives

$$H_* \simeq \pi M_{\rm Pl}\sqrt{\frac{r\mathcal{P}_{\mathcal{R}}(k_*)}{2}}. \tag{66}$$

Equivalently, the potential energy during inflation is

$$V_*^{1/4} \simeq \left(\frac{3\pi^2}{2}\mathcal{P}_{\mathcal{R}}(k_*)r\right)^{1/4} M_{\rm Pl}. \tag{67}$$

Numerically,

$$V_*^{1/4} \sim 10^{16}\,{\rm GeV}\left(\frac{r}{10^{-2}}\right)^{1/4}. \tag{68}$$

Example: monomial potentials

For a potential

$$V(\phi) = \lambda_p \phi^p, \tag{69}$$

the slow-roll predictions at leading order in $1/N_*$ are

$$n_s \simeq 1 - \frac{p+2}{2N_*}, \qquad r \simeq \frac{4p}{N_*}. \tag{70}$$

For $p = 2$ and $N_* = 60$,

$$n_s \simeq 0.967, \qquad r \simeq 0.13. \tag{71}$$

The value of n_s is reasonable, but the tensor amplitude is too large compared with current bounds.

Example: plateau potentials

Plateau models have potentials that become asymptotically flat at large field values. A representative example is the Starobinsky form

$$V(\phi) = \Lambda^4 \left(1 - e^{-\sqrt{2/3}\,\phi/M_{\rm Pl}}\right)^2. \tag{72}$$

The leading predictions are

$$n_s \simeq 1 - \frac{2}{N_*}, \qquad r \simeq \frac{12}{N_*^2}. \tag{73}$$

For $N_* = 60$,

$$n_s \simeq 0.967, \qquad r \simeq 0.003. \tag{74}$$

Plateau models illustrate how a red scalar tilt can coexist with a small tensor amplitude.

Example: natural inflation

Natural inflation, proposed by Freese, Frieman, and Olinto, uses a pseudo-Nambu–Goldstone boson as the inflaton. The approximate shift symmetry protects the flatness of the potential.

The potential is

$$V(\phi) = \Lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right) \right], \tag{75}$$

or equivalently with a shifted origin,

$$V(\phi) = \Lambda^4 \left[1 - \cos\left(\frac{\phi}{f}\right) \right]. \tag{76}$$

The slow-roll parameters are controlled by the ratio $f/M_{\rm Pl}$. Successful slow roll typically requires

$$f \gtrsim M_{\rm Pl}. \tag{77}$$

The inflaton is light because it is protected by an approximate global symmetry. The potential is generated only by small symmetry-breaking effects.

The Lyth bound

The tensor-to-scalar ratio is related to the inflaton field excursion. Since

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = M_{\mathrm{Pl}}\sqrt{2\epsilon_H} = M_{\mathrm{Pl}}\sqrt{\frac{r}{8}}, \tag{78}$$

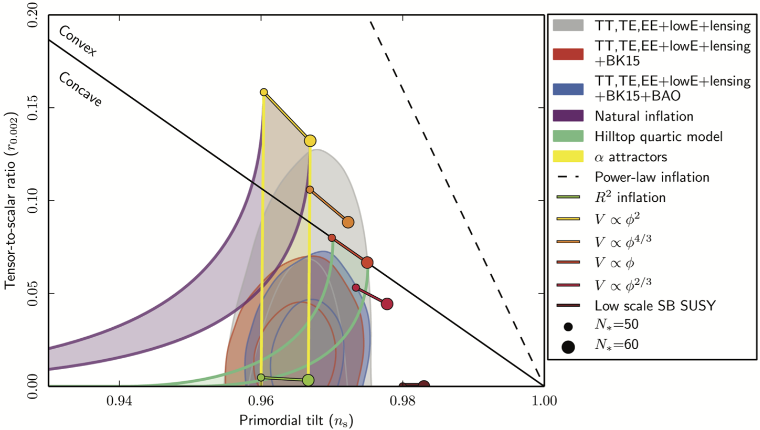
one obtains approximately

$$\frac{\Delta\phi}{M_{\mathrm{Pl}}} \gtrsim \mathcal{O}(1)\left(\frac{r}{10^{-2}}\right)^{1/2}\left(\frac{\Delta N}{50}\right). \tag{79}$$

Interpretation

A detectable tensor amplitude at the level $r \gtrsim 10^{-2}$ typically implies a super-Planckian field excursion in single-field slow-roll inflation.

Constraints in the (n_s, r) plane



Marginalized joint 68% and 95% CL regions for (n_s, r) at $k = 0.002\,\mathrm{Mpc}^{-1}$ from Planck, alone and combined with BK15 and BAO. Contours assume $\mathrm{d}n_s/\mathrm{d}\ln k = 0$.